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ESTIMATING THE TIME OF MOTION OF CERTAIN DYNAMICAL SYSTEMS†

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Developing results obtained earlier [1, 2], a method of changing Lyapunov's function with a negative-sign derivative into a Lyapunov function with a negative-definite derivative [1] is applied to natural mechanical systems with dissipation when there are no gyroscopic forces. The transition time is estimated. © 2002 Elsevier Science Ltd. All rights reserved.

1. CHANGING LYAPUNOV'S FUNCTION FOR ESTIMATING THE TRANSITION TIME

For the system of differential equations of disturbed motion

$$\dot{x}_i = f_i(x), \ x \in R^n, \ f_i(x) \in C^1(G), \ i = 1, 2, \dots, n \tag{1.1}$$

with an asymptotically stable equilibrium position $x = 0 \in G \subset R^n$, for which a positive-definite Lyapunov function $V_0(x)$ is known in the region G, with a non-positive time derivative by virtue of (1.1), which vanishes on the manifold $M \subset G$, a new function was constructed in [1]

$$V(x) = V_0(x) + V_*(x)$$

where

$$V_*(x) = \sum_{k=1}^{n} \lambda_k \Phi_k(x), \quad \Phi_k = -\int_0^{x_k} f_k^0(x_{m+1}, \dots, x_n) dx_k$$

$$f_k^0(x_{m+1}, \dots, x_n) = f_k(x_1^0(x_{m+1}, \dots, x_n), \dots, x_n)$$

$x_{m+1}, \dots, x_n$ are independent variables in terms of which the remaining variables $x_1, x_2, \dots, x_m$ on M are expressed in a unique and differentiable from: $x_j = x_j^0(x_{m+1}, \dots, x_n)$; $x_j^0(0) = 0 (j = 1, \dots, m)$ and λ_k are certain non-negative numbers.

From a consideration of the time derivative of V_* by virtue of system (1.1)

$$\dot{V}_* = -\sum_{i=1}^{m} \lambda_i f_i f_i^0 + \sum_{k=m+1}^{n} f_k \sum_{i=1}^{m} \lambda_i \frac{\partial \Phi_i}{\partial x_k} + \sum_{i,k=m+1}^{n} \lambda_k f_i \frac{\partial \Phi_k}{\partial x_i} \tag{1.2}$$

two cases of the selection of λ_i in which the problem of changing Lyapunov's function is solved were distinguished [1].

We will dwell on the case when

$$f_{m+1}(x) = f_{m+2}(x) = \dots f_n(x) = 0 \ \text{ on } \ M \tag{1.3}$$

This case arises in mechanical systems with energy dissipation when there are no gyroscopic forces. In fact, the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{d\dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = -\frac{\partial R}{d\dot{q}_i}$$

where

$$L = T - U, \quad R = \frac{1}{2}\sum_{i,k=1}^{n} \mu_{ik} \dot{q}_i \dot{q}_k, \quad T = T_2 + T_0, \ T_2 = \frac{1}{2}\sum_{i,j}^{n} a_{ij} \dot{q}_i \dot{q}_j$$

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(L is the Lagrange function, R is the Rayleigh function, μ_{ik} are the dissipation coefficients, and T_0, a_{ij} and U are functions of the generalized coordinates $q_1, \ldots, q_n$), after introducing. The Hamilton variables

$$p_k = \partial L / \partial \dot{q}_k \tag{1.4}$$

are transformed into

$$\dot{p}_k = \partial L / \partial q_k - \partial R / \partial \dot{q}_k \tag{1.5}$$

Taking into account that

$$L = \sum_i p_i \dot{q}_i - H$$

where $H = T + U$ is the Hamilton function, Eqs (1.5) can be given a different form

$$\dot{p}_k = -\partial H / \partial q_k - \partial R / \partial \dot{q}_k \quad \text{or} \quad \dot{p}_k = \sum_{i,j} A_{ij}^k p_i p_j + \sum_i A_i^k p_i + A_k \tag{1.6}$$

where A_{ij}^k, A_i^k and A_k are functions of the generalized coordinates, and here A_i^k are linear forms in μ_{kj}. These equations, together with the equations

$$\dot{q}_k = \sum_i B_i^k p_i; \; B_i^k = B_i^k(q) \tag{1.7}$$

obtained from expressions (1.4), form a closed system of equations.

Suppose the state of equilibrium

$$q_1 = \ldots = q_n = 0, \quad p_1 = \ldots = p_n = 0$$

is asymptotically stable, G is the region of attraction and $V_0 = T + U$ is Lyapunov's function. Then, expression (1.2) for $\dot{V}_*$ will have the form of the case considered here if x is thought of as a vector with the coordinates $p_1, \ldots, p_n, q_1, \ldots q_n$, and if account is taken of the fact that the manifold M is described by the system of equations $p_1 = \ldots = p_n = 0$. Here

$$f_{n+1}(x) = f_{n+2}(x) = \ldots = f_{2n}(x) = 0 \text{ on } M$$

In order for $\dot{V}_*$ to be negative on $M \backslash \{0\}$, it is sufficient to put $\lambda_{n+1} = \ldots = \lambda_{2n} = 0$ for any positive $\lambda_1, \ldots, \lambda_n$.

We will now consider the more general case (1.3) of system (1.1). Let G_{12} be the closure of the part of region G contained between the surfaces $V_0 = c_1$ and $V_0 = c_2$, $c_1 > c_2 > 0$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_m = \lambda$.

By virtue of the continuity of the functions V_0, $\dot{V}_0$, V_* and $\dot{V}_*$ in the region G_{12}, a fairly small number λ, for which the sum $V_0 + V_*$ will be positive in G_{12}, will be found. In this case it may turn out that the sum $[-(\dot{V}_0 + \dot{V}_*)]$ will also be positive in G_{12}. Having determined min $(-\dot{V}_0 - \dot{V}_*) = \upsilon$ in G_{12}, it is possible to give (see, for example, [3]) an upper limit of the time of motion of the system in G_{12}, i.e. to estimate the transition time. However, if the sum $[-(\dot{V}_0 + \dot{V}_*)]$ with the selected λ is not positive over the entire region G_{12}, then a certain neighbourhood of the manifold M should exist in which this sum is positive. Then, when λ is reduced further, this neighbourhood will expand and, at a certain value $\lambda = \lambda_{**}$, will cover the entire region G_{12}, i.e. estimation of the transition time is always possible.

By varying the values of $\lambda_1, \ldots, \lambda_m$ and dividing the region G_{12} into subregions by specifying the numbers c_3, c_4, $\ldots$, c_k ($c_2 < c_3 < c_4 < \ldots < c_k < c_1$), the estimate of the transition time can be improved.

Along with this estimate, a rougher estimate may be useful. This is based on the inequality (in all cases below, the operations min and max are conducted over the region G_{12}; summation is carried out from $k = 1$ to $k = m$)

$$\min(V_0 + V_*) = \min(V_0 - \lambda \sum \Phi_k) \geqslant c_2 - \lambda \max \left| \sum \Phi_k \right| > 0$$

and on replacing the check of the positiveness of $(-\dot{V}_0 - \dot{V}_*)$ in the region G_{12} with the chosen value

$$\lambda = \lambda_*, \; \lambda_* = c_2 / \max \left| \sum \Phi_k \right|$$

with a check of the positiveness of $(-\dot{V}_*)$ in the ε-neighbourhood of the set M (or, more precisely, in the set $M_\varepsilon = (x \in G_{12} \,|\, d(x, M) \leqslant \varepsilon)$, where ε is a small positive number, and $d(x, M)$ is the distance between points $x \in G_{12}$ and the manifold M) and with a check of the positiveness of $(-\dot{V}_0 - \dot{V}_* - \alpha)$ in $G_{12} \backslash M_\varepsilon$, where $\alpha = \min(-\dot{V}_*)$ on M_ε.

For a fairly small fixed value of ε as $\lambda \to 0$ it follows that $\alpha \to 0$ and an increase in min $(-\dot{V}_0 - \dot{V}_* - \alpha)$ in $G_{12} \backslash M_\varepsilon$ to the value $\beta > 0$, $\beta = \min(-\dot{V}_0)$. Therefore, $\lambda = \lambda_{*0} \leq \lambda_*$ will be found, for which $\alpha = \alpha_0 > 0$ and the quantity min $(-\dot{V}_0 - \dot{V}_* - \alpha_0)$ will become non-negative. However, the value of α_0 can be taken as the lower limit of the values of $(-\dot{V}_0 - \dot{V}_*)$ in the region G_{12} and used to estimate the transition time.

2. EXAMPLE: THE MOTION OF A SYSTEM OF TWO SERIES-CONNECTED PENDULUMS IN A RESISTING MEDIUM

Assuming that the resistance of the medium is proportional to the velocity of motion of each pendulum, expressions were obtained in [4] for the resistance of the medium to the motion of two series-connected pendulums

$$R_1 = \mu[(m_1 + m_2)l_1^2\dot{\varphi}_1 + m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1)\dot{\varphi}_2]$$
$$R_2 = \mu[m_2 l_2^2\dot{\varphi}_2 + m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1)\dot{\varphi}_1]$$

where μ is a positive constant, and m_k, l_k and φ_k are the mass, length and angle of deviation of the pendulums from the vertical ($k = 1, 2$).

The equations of motion of the system and its kinetic and potential energy have the form

$$\begin{aligned} &\dot{y}_1 = -\mu y_1 - \omega k_1 + \psi_1, \quad \dot{\varphi}_1 = y_1 \\ &\dot{y}_2 = -\mu y_2 - A\omega k_2 + \psi_2, \quad \dot{\varphi}_2 = y_2 \\ &T = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\varphi}_1^2 + \frac{1}{2}m_2 l_2\dot{\varphi}_2^2 + m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1)\dot{\varphi}_1\dot{\varphi}_2 \\ &U = (m_1 + m_2)g l_1(1 - \cos\varphi_1) + m_2 g l_2(1 - \cos\varphi_2) \end{aligned} \tag{2.1}$$

where

$$\omega = \frac{1}{B[A - b\cos^2(\varphi_2 - \varphi_1)]}, \quad A = \frac{(m_1 + m_2)l_1}{m_2 l_2}, \quad B = m_2 l_2, \quad b = \frac{l_1}{l_2}$$
$$k_1 = (m_1 + m_2)g\sin\varphi_1, \quad k_2 = m_2 g\sin\varphi_2$$
$$\psi_1 = \omega\cos(\varphi_2 - \varphi_1)k_2 + B\omega\sin(\varphi_2 - \varphi_1)[b\cos(\varphi_2 - \varphi_1)y_1^2 + y_2^2]$$
$$\psi_2 = b\omega\cos(\varphi_2 - \varphi_1)k_1 - Bb\sin(\varphi_2 - \varphi_1)[Ay_1^2 + \cos(\varphi_2 - \varphi_1)y_2^2]$$

and g is the acceleration due to gravity.

We will take the following as Lyapunov's function [4]:

$$V_0 = ABby_1^2 + By_2^2 + 2Bby_1y_2\cos(\varphi_2 - \varphi_1) + 2b\int_0^{\varphi_1} k_1(x)dx + 2b\int_0^{\varphi_2} k_2(x)dx \tag{2.2}$$

Its time derivative, by virtue of system (2.1), is

$$\dot{V}_0 = -2\mu B\{b[A - b\cos^2(\varphi_2 - \varphi_1)]y_1^2 + [by_1\cos(\varphi_2 - \varphi_1) + y_2]^2\} < 0$$

for $y_1 \neq 0$ and $y_2 \neq 0$. Thus, the manifold M is represented by the system of equations

$$y_1 = 0, \quad y_2 = 0$$

and

$$f_1^0 = \omega[-k_1 + \cos(\varphi_2 - \varphi_1)k_2], \quad f_2^0 = \omega[-Ak_2 + b\cos(\varphi_2 - \varphi_1)k_1]$$
$$f_3^0 \equiv 0, \quad f_4^0 \equiv 0, \quad \Phi_1 = -f_1^0 y_1, \quad \Phi_2 = -f_2^0 y_2$$
$$\dot{V}_* = -\lambda(f_1 f_1^0 + f_2 f_2^0 + f_3 f_1^0 + f_4 f_2^0)$$

According to (1.8)

$$\lambda_* = c_2 / \max|f_1^0 y_1 + f_2^0 y_2|$$

where

$$|f_1^0 y_1 + f_2^0 y_2| < \frac{g}{l_1 m_1}[(m_1+m_2)(1+b)+m_2(1+A)]\max(|y_1, y_2|)$$

$$\max(|y_1|,|y_2|) \leq l$$

and l is the major semiaxis of the ellipse $Aby_1^2 - y_2^2 - 2by_1y_2 = c_1$.

We will check the positiveness of $(-\dot{V}_*)$ in M_ε, assuming ε to be a fairly small number. From the expression for $\dot{V}_*$ it can be seen that the sign of $\dot{V}_*$ in M_ε may depend only on the choice of ε. When $\varepsilon \to 0$ we have $y_1 \to 0$ and $y_2 \to 0$. For a fairly small value $\varepsilon = \varepsilon_*$, we will have $(-\dot{V}_*) \geq \alpha > 0$ in M_ε, and it is possible to determine α_0.

3. THE MOTION OF A PARTICLE IN A CENTRAL FIELD OF FORCES TAKING THE RESISTANCE OF THE MEDIUM INTO ACCOUNT

Consider a particle of unit mass moving in a medium in which the friction depends on the velocity υ in the plane Oxy with a centre of attraction at the point O. Let $T = \upsilon^2/2$ be the kinetic energy, where $\upsilon = \sqrt{\dot{x}^2 + \dot{y}^2}$.

The equations of motion of the particle have the form (the prime denotes a derivative with respect to r)

$$\dot{p}_1 = -\frac{x}{r}U' - \frac{p_1}{p}Q, \quad \dot{p}_2 = -\frac{y}{r}U' - \frac{p_2}{p}Q, \quad \dot{x} = p_1, \quad \dot{y} = p_2 \tag{3.1}$$

$$p = \sqrt{p_1^2 + p_2^2}, \quad r = \sqrt{x^2 + y^2}$$

where $Q = Q(p)$ is the friction force, $U = U(r) > 0$ is the potential energy, and $U' > 0$ where $r > 0$.

The origin of coordinates of the phase space is a global attractor [3]. We will assume that $V_0 = T(\upsilon) + U(r)$. In accordance with Section 1, we calculate

$$\dot{V}_0 = -Q(p)p \quad (\dot{V}_0 \leq 0)$$

$$f_1^0 = -\frac{x}{r}U', \quad f_2^0 = -\frac{y}{r}U', \quad f_3^0 = 0, \quad f_4^0 = 0$$

$$\Phi_1 = \frac{x}{r}U'p_1, \quad \Phi_2 = \frac{y}{r}U'p_2$$

$$V_* = \frac{\lambda}{r}U'(xp_1 + yp_2)$$

$$\dot{V}_* = -\lambda(U')^2 + \lambda\left[\frac{1}{r^2}\left(\frac{y_2}{r}U' + x^2U''\right)p_1^2 + 2p_1p_2\frac{xy}{r^2}\left(U'' - \frac{1}{r}U'\right) + \right.$$
$$\left. + \frac{1}{r^2}\left(\frac{x^2}{r}U' + y^2U''\right)p_2^2 - \frac{Q}{pr}(xp_1 + yp_2)U'\right]$$

In particular, when

$$U(r) = gr^k, \quad k \geq 2, \quad g > 0; \quad Q = \mu p, \quad \mu > 0$$

we obtain

$$V = V_0 + V_* = \tfrac{1}{2}(p_1^2 + p_2^2) + gr^k + \lambda kgr^{k-2}(xp_1 + yp_2)$$

$$\dot{V} = -\mu(p_1^2 + p_2^2) - \lambda k^2(k-1)^2 r^{2k-4} +$$
$$+ \lambda gkr^{k-4}[(y^2 + x^2(k-1))p_1^2 + 2(k-2)xyp_1p_2 +$$
$$+ (x^2 + y^2(k-1))p_2^2 - \mu r^2(xp_1 + yp_2)]$$

$$\lambda_* = c_2 / \max|gkr^{k-2}(xp_1 + yp_2)|$$

Taking into account that

$$gr^k \leq c_1; \quad |xp_1 + yp_2| \leq r\max|p_1 + p_2|$$

$$p_1^2 + p_2^2 \leq 2c_1; \quad |p_1 + p_2| \leq \sqrt{2(p_1^2 + p_2^2)} \leq 2\sqrt{c_1}$$

we obtain

$$\lambda_* \geqslant c_2/(2g^{1/k}kc_1^{-1/k+3/2}) \tag{3.2}$$

We will determine the set M_ε, i.e. find ε for which

$$-\lambda_*^{-1}\dot{V}_* \geqslant 0, \quad \text{if} \quad |p_1|<\varepsilon,\ |p_2|<\varepsilon$$

Since

$$\tfrac{1}{2}(p_1^2+p_2^2)+gr^k \geqslant c_2$$

then

$$r \geqslant ((c_2-\varepsilon^2)/g)^{1/k}, \quad (\varepsilon^2<c_2)$$

and

$$-\lambda_*^{-1}\dot{V}_* \geqslant k^2(k-1)^2g^{4/k-2}(c_2-\varepsilon^2)^{2-4/k}-$$
$$-2kg^{2/k}c_1^{1-2/k}\left(k-1+(k-2)\frac{c_1}{g}\right)\varepsilon^2-\sqrt{2}\mu g^{1/k}kc_1^{1-1/k}\varepsilon \geqslant 0 \tag{3.3}$$

Let ε_1 be the positive solution of the equation that is closest to zero, corresponding to the latter inequality. Then

$$-\dot{V}_* \geqslant \lambda_*\alpha(\varepsilon)>0$$

where $\alpha(\varepsilon)$ denotes the left-hand side of inequality (3.3) when $\varepsilon = \Theta_1\varepsilon_1$ and $0 < \Theta_1 < 1$, and λ_* is the right-hand side of inequality (3.2).

We will now estimate λ for which $\dot{V}_0 + \dot{V}_* < 0$ in the region $G_{12}\backslash M_\varepsilon$. In the given region ($\varepsilon < |p_1| < \sqrt{c_1}$; $\varepsilon < |p_2| < \sqrt{c_1}$)

$$\dot{V}_* = 2\lambda kg^{1/k}c_1^{2-2/k}[2(k-1)g^{1/k}+\mu c_1^{-1/2+1/k}]$$
$$\dot{V}_0 < -\mu(p_1^2+p_2^2) \leqslant -2\mu\varepsilon^2$$

In order for the sum of the left-hand sides of the latter inequalities to be negative, it is sufficient to satisfy the following inequality

$$\lambda \leqslant \lambda_{**} = \mu\varepsilon^2/\{kg^{1/k}c_1^{2-2k}[2(k-1)g^{1/k}+\mu c_1^{1/k-1/2}]\}$$

Assuming that

$$\lambda_1 = \Theta_2\lambda_{**} \leqslant \lambda_*, \quad 0<\Theta_2<1$$

we obtain

$$\dot{V} \leqslant -\min(2\mu\varepsilon^2(1-\Theta_2),\ \lambda_1\alpha)$$

Remark. If, for the given dynamical system, two Lyapunov functions are known, differing in that their derivatives vanish on different manifolds, then, by combining these functions, it is possible to obtain a new function with a negative-definite derivative. For example, for the equation

$$\ddot{x}+\varphi(x)\dot{x}+f(x)=0, \quad \varphi(x)>0, \quad xf(x)>0, \quad x\neq 0$$

the Liénard replacement [4]

$$y=\dot{x}+\Phi(x), \quad \Phi(x)=\int_0^x \varphi(x)dx$$

leads to the system

$$\dot{x}=y-\Phi(x); \quad \dot{y}=-f(x)$$

with Lyapunov function and its derivative

$$V_1 = y^2 + 2F(x), \quad F(x) = \int_0^x f(x)dx$$

$$\dot{V}_1 = -2f(x)\Phi(x)$$

which vanishes when $x = 0$. However, it is also possible to consider the equivalent system

$$\dot{x} = z, \quad \dot{z} = -\varphi(x)z - f(x)$$

with Lyapunov function

$$V_2 = z^2 + 2F(x)$$

and its derivative

$$\dot{V}_2 = -2\varphi(x)z^2$$

After reduction to the variables $x, y (z = y - \Phi(x))$, we obtain

$$V = V_1 + V_2 = y^2 + 4F(x) + (y - \Phi(x))^2$$

$$\dot{V} = -2f(x)\Phi(x) - 2\varphi(x)(y - \Phi(x))^2$$

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